Holomorphic vector fields and minimal Lagrangian submanifolds

Edward Goldstein

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Abstract

The purpose of this note is to establish the following theorem: Let N be a Kahler manifold, L be an oriented immersed minimal Lagrangian submanifold of N without boundary and V be a holomorphic vector field in a neighbourhood of L in N. Let $div(V)$ be the (complex) divergence of V. Then the integral $\int_L div(V) = 0$. Vice versa suppose that N^{2n} is Kahler-Einstein with non-zero scalar curvature and L is an embedded totally real n-dimensional oriented real-analytic submanifold of N s.t. for any holomorphic vector field V defined in a neighbourhood of L in N, $\int_L div(V) = 0$. Then L is a minimal Lagrangian submanifold of N.

1 Basic properties

Let (N^{2n}, ω) be a Kahler manifold. In this section we will discuss holomorphic vector fields on N and present some basic facts on minimal Lagrangian submanifolds of N. The results of this section are essentially known, though they might be stated in different terms in the literature.

First we discuss holomorphic vector fields on N. Let V be a vector field defined on some open subset U of N. The following proposition is elementary and well known (see [2], Proposition 4.1):

Proposition 1 *The following conditions are equivalent:*

1) The flow of V commutes with the complex structure J on N.

2) For any point $m \in U$ the following endomorphism : $X \mapsto \nabla_X V$ of $T_m N$ is J-linear on $T_m N$.

3) The vector field $V - iJV$ gives a holomorphic section of $T^{(1,0)}U$.

A vector field V satisfying the conditions of Proposition 1 is called a *holomorphic* vector field. Let V be a holomorphic vector field on some open subset U of N and let m be a point in U. Since the endomorphism $X \mapsto \nabla_X V$ is J-linear on $T_m N$ we can define

$$div(V) = trace_{\mathbb{C}}(X \mapsto \nabla_X V) \tag{1}$$

Let $f = Ref + iImf$ be a holomorphic function on U. From condition 3) of Proposition 1 we deduce that the vector field $fV = RefV + ImfJV$ is a holomorphic vector field on U. Moreover one easily computes that

$$div(fV) = fdiv(V) + V(f) \tag{2}$$

Let $K(N)$ be the canonical bundle of N (i.e. $K(N) = \Lambda^{(n,0)}T^*N$). Let φ be a section of $K(N)$ over U (not necessarily holomorphic). Thus φ is an $(n,0)$-form over U.

Proposition 2 *Let V be a holomorphic vector field on U. Then*

$$\nabla_V\varphi = \mathcal{L}_V\varphi - div(V)\varphi$$

Proof: Let $m \in U$. Pick a unitary basis $X_1, \ldots, X_n$ of T_mN (here T_mN is viewed as Hermitian vector space with the complex structure J). Extend X_i to a unitary frame in a neighbourhood of m. Then

$$\mathcal{L}_V\varphi(X_1, \ldots, X_n) = V(\varphi(X_1, \ldots, X_n)) - \Sigma\varphi(X_1, \ldots, [V, X_n], \ldots, X_n) \tag{3}$$

and

$$\nabla_V\varphi(X_1, \ldots, X_n) = V(\varphi(X_1, \ldots, X_n)) - \Sigma\varphi(X_1, \ldots, \nabla_V X_i, \ldots, X_n) \tag{4}$$

Now $\nabla_V X_i = [V, X_i] + \nabla_{X_i}V$. We plug this into (4) and subtract (4) from (3) to deduce the statement of the proposition. Q.E.D.

Next we prove the following lemma (which we essentially proved in [1]):

Lemma 1 *Let (N, ω) be a Kahler-Einstein manifold with non-zero scalar curvature t. Let V be a holomorphic infinitesimal isometry on some neighbourhood U of N. Then the function $\mu = it^{-1}div(V)$ is a moment map for the V-action on (N, ω)*

Proof: We need to prove that $d\mu = i_V\omega$. We shall prove it at a point m s.t. $V(m) \neq 0$. Pick an element φ of $K(N)$ over m which has unit length. Since the flow of V is given by holomorphic isometries we can extent φ to a unit length section of $K(N)$ invariant under the V-flow on some neighbourhood U of m. The section φ defines a connection 1-form ξ on U, $\xi(u) =< \nabla_u\varphi, \varphi >$. The Einstein condition tells that

$$id\xi = t\omega \tag{5}$$

Since φ is V-invariant we deduce from Proposition 2 that

$$div(V) = -\xi(V) \tag{6}$$

Also since φ is V-invariant and the flow of V is given by isometries, we deduce that ξ is also V-invariant. Thus

$$0 = \mathcal{L}_V\xi = d(\xi(V)) + i_V d\xi = \text{ by (5) and (6)} = -d(div(V)) - iti_V\omega$$

and the lemma follows. Q.E.D.

Next we discuss minimal Lagrangian submanifolds on N. Let L be an oriented n-dimensional totally real submanifold of N (i.e. $TL \bigcap J(TL) = 0$). For any point $l \in L$ there is a unique element κ_l of $K(N)$ over l which restricts to the volume form on L. Various k_l give rise to a section

$$\kappa : L \mapsto K(N) \tag{7}$$

Let now L be a Lagrangian submanifold of N. The section κ is a unit length section of $K(N)$ over L and it defines a connection 1-form ξ for the connection on $K(N)$ over L, $\xi(u) =< \nabla_u \kappa, \kappa >$. Here ∇ is the connection on $K(N)$, induced from the Levi-Civita connection on N. Since κ has unit length ξ is an imaginary valued 1-form.

Let h be the trace of the second fundamental form of L. So h is a section of the normal bundle of L in N and we have a corresponding 1-form $\sigma = i_h\omega$ on L. The following fact is well-known, although it is often stated differently in the literature (see [3]):

Lemma 2 $\sigma = i\xi$

Proof: Let $l \in L$ and e be some vector in the tangent space to L at L. To compute $\xi(e)$ we need to compute $\nabla_e \kappa$. Take an orthonormal frame (v_j) of $T_l L$ and extend it to an orthonormal frame in a neighbourhood U of l in L s.t. $\nabla^L v_i = 0$ at l (here ∇^L is the Levi-Civita connection of L). We get that

$$\nabla_e \kappa = \kappa \cdot \nabla_e \kappa(v_1, \ldots, v_n) = \kappa(e(\kappa(v_1, \ldots, v_n)) - \Sigma\kappa(v_1, \ldots, \nabla_e v_j, \ldots, v_n))$$

Now $e(\kappa(v_1, \ldots, v_n)) = 0$. Also clearly

$$\kappa(v_1, \ldots, \nabla_e v_j, \ldots, v_n) = i < \nabla_e v_j, Jv_j >= i < \nabla_{v_j} e, Jv_j >= i < -e, J(\nabla_{v_j} v_j) >$$

Here J is the complex structure on N. Thus we get that

$$\nabla_e \kappa = -i(Jh \cdot e)\kappa_l = -i\sigma(e)\kappa_l$$

Here $h = \Sigma \nabla_{v_j} v_j$ is the trace of the second fundamental form of L. Thus $\sigma = i\xi$. Q.E.D.

Thus if L is *minimal* (i.e. $h = 0$) iff κ is parallel over L.

Remark: Let L be a minimal Lagrangian submanifold of N. We have seen that $\xi = 0$ on L. Thus also $d\xi = 0$ on L. But $d\xi$ is the curvature form for the connection on $K(N)$, i.e. $d\xi = -iRic$. Here Ric is the Ricci form of N, and Ric is proportional to ω iff N is Kahler-Einstein. If N is Kahler-Einstein the condition $Ric|_L = 0$ follows from the Lagrangian condition on L. But if N is not Kahler-Einstein, we have a new algebraic condition $Ric|_L = 0$ on minimal Lagrangian submanifolds.

2 Proof of the main theorem

We now can state and prove our main theorem:

Theorem 1 *1) Let N be a Kahler manifold, L be an oriented immersed minimal Lagrangian submanifold of N without boundary and V be a holomorphic vector field defined in a neighbourhood of L in N. Then*

$$\int_L div(V) = 0$$

2) Let N^{2n} be a Kahler-Einstein manifold with non-zero scalar curvature and L be an n-dimensional totally real oriented embedded real-analytic submanifold of N s.t. for any holomorphic vector field V defined in a neighbourhood of L in N we have $\int_L div(V) = 0$. Then L is a minimal Lagrangian submanifold of N

Proof: 1) Let L be a minimal Lagrangian submanifold of N and V be a holomorphic vector field defined in a neighbourhood of L in N. Let κ be a section of $K(N)$ over L as in equation (7). Since κ restricts to the volume form on L we have $\int_L div(V) = \int_L div(V)\kappa$. Let

$$\phi = i_V \kappa|_L \tag{8}$$

ϕ is an $(n-1)$-form on L. We claim that

$$d\phi = div(V)\kappa|_L \tag{9}$$

Thus the first assertion of the theorem will follow. To prove (9) let l be a point in L. By Lemma 2 we have that for any element w in the tangent bundle to L, $\nabla_w \kappa = 0$. We can extend κ to be a section of $K(N)$ over some neighbourhood Z of l in N s.t. for any element w in the normal bundle of L to N in $Z \bigcap L$ we'll have $\nabla_w \kappa = 0$. Thus we'll have $\nabla \kappa = 0$ along L. From this it also follows that $d\kappa = 0$ along L. Now we use equation Proposition 2 for V and $\varphi = \kappa$. We deduce that

$$div(V)\kappa = \mathcal{L}_V \kappa$$

along L. Also $\mathcal{L}_V \kappa = d(i_V \kappa) + i_V(d\kappa)$ and $d\kappa$ vanishes along L. Thus we get

$$div(V)\kappa|_L = d\phi$$

2) Let (N^{2n}, ω) be a Kahler-Einstein manifold with a non-zero scalar curvature t and L be an oriented real-analytic embedded totally real n-dimensional submanifold of N s.t. $\int_L div(V) = 0$ for any holomorphic vector field V near L. Consider the section κ of $K(N)$ over L as in (7). Since L is totally real, n-dimensional embedded real-analytic submanifold of N, one can uniquely extend κ to a holomorphic section of $K(N)$ over some neighbourhood U' of L in N (see the Appendix).

Let $\xi = Re\xi + iIm\xi$ be the connection 1-form on L defined by the section κ over L, i.e. for any tangent vector u to L we have $\nabla_u \kappa = \xi(u)\kappa$. Let V_r be the

vector field on L dual to the form $Re\xi$ with respect to the Riemannian metric on L. V_r is a real-analytic vector field on L and by Proposition 4 of the Appendix we can extend V_r to a holomorphic vector field V_r on a neighbourhood of L in N. By Proposition 2 we have

$$\xi(V_r)\kappa = \mathcal{L}_{V_r}\kappa - div(V_r)\kappa \ on \ L$$

We integrate this over L to get

$$\int_L \xi(V_r)\kappa = \int_L \mathcal{L}_{V_r}\kappa - \int_L div(V_r)\kappa \tag{10}$$

Now $\mathcal{L}_{V_r}\kappa = d(i_{V_r}\kappa)$ and it integrates to 0 over L. Also by our assumptions since κ restricts to the volume form on L we get

$$\int_L div(V_r)\kappa = \int_L div(V_r) = 0$$

Thus $\int_L \xi(V_r)\kappa = 0$. But $Re(\xi(V_r)) = |V_r|^2$ pointwise. Thus $V_r = 0$ and so $Re\xi = 0$. Similarly we prove that $Im\xi = 0$.

Thus $\xi = 0$ on L. So $d\xi = 0$ on L. But

$$d\xi = -it\omega|_L$$

Here ω is the Kahler form on N. Hence L is Lagrangian. Since $\xi = 0$ on L we deduce from Lemma 2 that L is minimal. Q.E.D.

Let us derive a simple corollary of Theorem 1:

Corollary 1 *Let L be an immersed oriented minimal Lagrangian submanifold of $\mathbb{C}P^n$ and let $(z_1, \ldots, z_{n+1})$ be homogeneous coordinates on $\mathbb{C}P^n$. Then we can't have $|z_1| > |z_2|$ at all points of L.*

Proof: Consider the following circle action on $\mathbb{C}P^n$:

$$e^{i\theta}(z_1, \ldots, z_{n+1}) = (e^{i\theta}z_1, e^{-i\theta}z_2, z_3, \ldots, z_{n+1})$$

Let V be the vector field on $\mathbb{C}P^n$ generating this action. $\mathbb{C}P^n$ is Kahler-Einstein with scalar curvature 1, hence by Lemma 1 the function $idiv(V)$ is a moment map for the S^1-action on $\mathbb{C}P^n$. We have computed in [1] that

$$idiv(V) = (|z_1|^2 - |z_2|^2)/\Sigma|z_i|^2$$

In fact we can also deduce this from Theorem 1. Indeed the map $f = (|z_1|^2 - |z_2|^2)/\Sigma|z_i|^2$ is a moment map for the S^1-action on $\mathbb{C}P^n$, hence it differs from $idiv(V)$ by a constant c. Also the submanifold $L' = ((z_1, \ldots, z_{n+1})||z_1| = |z_j|)$ is a minimal Lagrangian submanifold of $\mathbb{C}P^n$. Hence by Theorem 1 $\int_{L'} div(V) = 0$. From this we deduce that $c = 0$ i.e. $idiv(V) = f$.

Let now L be an immersed oriented minimal Lagrangian submanifold of $\mathbb{C}P^n$. We have $\int_L div(V) = 0$. Hence we obviously can't have $|z_1| > |z_2|$ everywhere on L. Q.E.D.

3 Appendix

In this Appendix we want to demonstrate the following fact (used in the proof of Theorem 1): Let L be a totally real n-dimensional embedded real-analytic compact submanifold of a complex manifold N^{2n}. Suppose that P is a holomorphic vector bundle over N and we have a real-analytic section σ of P over L. Then we can uniquely extend it to a holomorphic section σ' over some neighbourhood of L in N. We begin with the following proposition:

Proposition 3 *Let $f : U \mapsto \mathbb{C}^k$ be a real analytic map from an open subset U of 0 in $\mathbb{R}^n$ to $\mathbb{C}^k$. Then we can uniquely extend f to a holomorphic map f' from some open subset U' of 0 in $\mathbb{C}^n$ to $\mathbb{C}^k$. Here we think of $\mathbb{R}^n$ as a subset of $\mathbb{C}^n$.*

Proof: Let $(z_1, \ldots, z_k)$ be coordinates on $\mathbb{C}^k$. We can think of f as

$$f = (f_1, \ldots, f_k)$$

and we need to extend each f_i to a holomorphic function on an open subset of 0 in $\mathbb{C}^n$. Let $x = (x_1, \ldots, x_n)$ be the coordinates on $\mathbb{R}^n$. Since f_i is real-analytic on $\mathbb{R}^n$ we can write its Taylor's expansion

$$f_i = \Sigma C_\alpha x^\alpha$$

near $0 \in \mathbb{R}^n$. Clearly f_i has a unique holomorphic extension

$$f_i' = \Sigma C_\alpha z^\alpha$$

onto a neighbourhood of 0 in $\mathbb{C}^n$. Q.E.D.

Now we can prove the main result of the Appendix:

Proposition 4 *Let L be a totally-real n-dimensional embedded compact real-analytic submanifold of a complex manifold N. Suppose that P is a holomorphic vector bundle over N and we have a real-analytic section σ of P over L. Then σ extends uniquely to a holomorphic section σ' on a neighbourhood of L is N*

Proof: It is obviously enough to prove that for any point $l \in L$ we can uniquely extend σ onto a neighbourhood of l in N. Also near l we can think of P as being the trivial bundle $\mathbb{C}^k$. Suppose now that there is a biholomorphic map ϕ from a neighbourhood U of 0 in $\mathbb{C}^n$ onto a neighbourhood U' of l in N s.t. $\phi(\mathbb{R}^n \bigcap U) = L \bigcap U'$. Then the desired claim will follow from Proposition 3.

To construct the biholomorphic map ϕ we again use Proposition 3. Since L is a real-analytic submanifold of N we can find a neighbourhood W of 0 in $\mathbb{R}^n$, a neighbourhood W' of l in L and a real-analytic map $f : W \mapsto N$ s.t. the image of f lies in L and in fact $f : W \mapsto W'$ is a diffeomorphism. Since N is a complex manifold we can find a neighbourhood of l in N, which is biholomorphic to a ball in $\mathbb{C}^n$. Thus we can think of f as a map form W to $\mathbb{C}^n$. By Proposition 3 we can extend it to a holomorphic map $f' = \phi$ from a neighbourhood of 0 in $\mathbb{C}^n$ to $\mathbb{C}^n$. Since $f : W \mapsto W'$ is a diffeomorphism it is clear that the differential of ϕ at $0 \in \mathbb{C}^n$ is an isomorphism. Thus ϕ is a biholomorphic map from some neighbourhood U of 0 in $\mathbb{C}^n$ and we are done. Q.E.D.

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Massachusetts Institute of Technology
E-Mail : egold@math.mit.edu